Instructor: **Yaroslav Korobka** Updated: October 13, 2025
Course: **Mathematics**

1 Basic topology

1.1 Metric spaces

Definition 1.1. A set X, whose elements we shall call *points*, is said to be *a metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the *distance* from p to q, such that

(a) $d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$,

(b) $d(p,q) = d(q,p)$,

(c) $d(p,q) \leq d(p,r) + d(r,q)$, for $\forall r \in X$.

Any function with these three properties is called a *distance function*, or *a metric*.

Example 1.2 (Metric spaces)**.** The following are examples of the metric spaces:

1. the set of real numbers $\mathbb{R}$ with a metric $d(p,q) = |p-q|$,
2. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = \sqrt{(p_1-q_1)^2 + (p_2-q_2)^2} := \|\mathbf{p}-\mathbf{q}\|$ (Eucledian distance),
3. a real plane $\mathbb{R}^2$ with a metric $d(\mathbf{p},\mathbf{q}) = |p_1-q_1| + |p_2-q_2|$ (Manhattan distance),
4. the set of probability distributions defined on the same measurable space with a metric $d(P,Q) = \frac{1}{\sqrt{2}}\left(\int \left(\sqrt{p(x)} - \sqrt{q(x)}\right)^2 dx\right)^{1/2}$ (Hellinger distance).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. Thus, every subset of a Euclidean space is a metric space.

Definition 1.3. By the *segment* (a,b) we mean the set of all real numbers x such that $a < x < b$. By the *interval* $[a,b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

If $a_i < b_i$ for $i = 1,\ldots,k$, the set of all points $\mathbf{x} = (x_1,\ldots,x_k)$ in $\mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ ($1 \leq i \leq k$) is called a *k-cell*. Thus, a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at $\mathbf{x}$ and radius r is defined to be the set of all $\mathbf{y} \in \mathbb{R}^k$ such that $\|\mathbf{y}-\mathbf{x}\| < r$ (or $\|\mathbf{y}-\mathbf{x}\| \leq r$).

We call a set $E \subset \mathbb{R}^k$ *convex* if

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$. For example, balls are convex. It is also easy to see that k-cells are convex.

Definition 1.4. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) *A neighborhood* of a point p is a set $N_r(p)$ consisting of all points q such that $d(p,q) < r$. The number r is called the *radius* of $N_r(p)$.

(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$. Example: take a set $A := (0,1)$. Point 0 is a limit point, because any open interval, say $(-\varepsilon, \varepsilon)$, intersects A.

(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E. Example: take a set $A = \{n^{-1} : n \in \mathbb{N}\}$. Each element is an isolated point because you can take a small interval around n^{-1} that avoids the other fractions in the set.

(d) E is *closed* if every limit point of E is a point of E. Example: take $A = [0,1]$. Both 0 and 1 are limit points and both belong to the set A. A set $B = (0,1]$ is not closed because a limit point 0 does not belong to the set.

(e) A point p is an *interior point* of E if there is a neighborhood $N_r(p)$ of p such that $N \subset E$. Example: take a set $A = (0,1)$. A point 0.5 is an interior point because there is a neighborhood around it, say, $N_{0.1}(0.5)$ that belongs to the set A; if $N_{0.1}(0.5) = (0.4, 0.6) := B$, we have $B \subset A$. On the other hand, if $C = [0.5, 1]$, 0.5 is not an interior point of C, because there is no neighborhood around it that is a subset of C; some points of that neighborhood are outside of C.

(f) E is *open* if every point of E is an interior point of E.

(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.

(h) E is *perfect* if E is closed and if every point of E is a limit point of E. Example: take $A = [0,1]$, which is closed with all points being limit points, so it is perfect. On the other hand, $B = [0,1] \cup \{3\}$ is not perfect because it contains a point 3, which is not a limit point (it is an isolated point).

(i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for $\forall p \in E$.

(j) E is *dense in* X if every point of X is a limit point of E, or a point of E (or both).

Let us note that in $\mathbb{R}^1$ neighborhoods are segments, whereas in $\mathbb{R}^2$ neighborhoods are interiors of circles.

Theorem 1.5. *Every neighborhood is an open set.*

Proof. Consider neighborhood $E = N_r(p)$, and let q be any point of E. Then there is a positive real number h such that

$$d(p,q) = r - h.$$

For all points s such that $d(q,s) < h$, we have then

$$d(p,s) \leq d(p,q) + d(q,s) < r - h + h = r,$$

so that $s \in E$. Thus, q is an interior point of E. □

Theorem 1.6. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.*

Proof. Suppose there is a neighborhood N of p which contains only a finite number of points of E. Let $q_1, \ldots, q_n$ be those points of $N \cap E$, which are distinct from p, and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E. This contradiction established the theorem. □

Corollary 1.7. *A finite point set has no limit points.*

Theorem 1.8. *A set E is open if and only if its complement is closed.*

1.2 Compact sets

Definition 1.9. By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.

Definition 1.10. A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite subcover*. More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K, then there are finitely many indices $\alpha_1, \ldots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}.$$

Corollary 1.11. *A set E is compact if it is both closed and bounded.*

1.3 Functions

Definition 1.12. Consider two sets A and B, whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B, which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* from A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f. The set of *all* values of f is called the *range* of f.

Definition 1.13. If for every $y \in B$ there is at most one $x \in A : f(x) = y$, the function f is said to be a 1-1 (*one-to-one*) mapping of A into B. This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

Definition 1.14. Let A and B be two sets and let f be a mapping of A into B. If $f(A) = B$, we say that f maps A *onto* B. If, additionally, f is 1-1, then f is *one-to-one and onto (bijection)*.

Definition 1.15. If there exists a 1-1 mapping of A *onto* B, we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$.

Definition 1.16. For any positive integer n, let J_n be the set whose elements are the integers $1, 2, \ldots, n$; let J be the set consisting of all positive integers. For any set A, we say:

(a) A is finite if $A \sim J_n$ for some n.

(b) A is *infinite* if A is not finite.

(c) A is *countable* if $A \sim J$.

(d) A is *uncountable* if A is neither finite nor countable.

(e) A is *at most countable* if A is finite or countable.

For two finite sets A and B, we evidently have $A \sim B$ if and only if A and B contain the same number of elements (same *cardinality*). For infinite sets, however, the idea of cardinality becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

Example 1.17. Let A be the set of all integers. Then A is countable. Consider, the following arrangement of the sets A and J:

$$\begin{aligned} A: &\quad 0, 1, -1, 2, -2, \ldots \\ J: &\quad 1, 2, 3, 4, 5, \ldots \end{aligned}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Remark 1.18. A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 1.17, in which J is a proper subset of A.

Definition 1.19. In the following, assume that the set A is a subset of $\mathbb{R}$.

(a) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \geq y$, then the set A is *bounded from above*.

(b) If there exists $x \in \mathbb{R}$ such that for every $y \in A$ we have $x \leq y$, then the set A is *bounded from below*.

(c) The *supremum* of A, denoted as $\sup A$, is the smallest upper bound of the set A.

(d) The *infimum* of A, denoted as $\inf A$, is the largest lower bound of the set A.

We note that the set A is bounded, if it is bounded both from below and from above, which is equivalent to the Definition 1.4(i). If the set A is not bounded from above, then $\sup A = \infty$, and if it is not bounded from below, then $\inf A = -\infty$.

2 Sequences and limits

Definition 2.1. By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$ for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by $x_1, x_2, x_3, \ldots$. The values of f, that is, the elements x_n, are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A*, or a *sequence of elements of A*.

Note that the terms $x_1, x_2, x_3, \ldots$ of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J, we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence".

Definition 2.2. For a given sequence $\{x_n\}$, if $x_{n+1} > x_n$ for $\forall n \in J$, then the sequence is *increasing*. If $x_{n+1} < x_n$ for $\forall n \in J$, then the sequence is *decreasing*. If $x_{n+1} \geq x_n$ for $\forall n \in J$, then the sequence is *non-decreasing*. If $x_{n+1} \leq x_n$ for $\forall n \in J$, then the sequence is *non-increasing*.

If at least one of these four conditions is satisfied, the sequence is called *monotonic*.

Example 2.3. We give examples of different sequences below.

(a) A sequence that is defined via a formula for the nth term: $x_n = \left(\frac{2}{3}\right)^n$.

(b) A sequence that is defined recursively (Fibonacci sequence): $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$, and $x_1 = x_2 = 1$.

(c) A sequence $x_n = (-1)^n$.

(d) A sequence $x_n = 2^n$.

Note that the sequence (a) is decreasing with n, while the sequence (b) is non-decreasing with n. The sequence (c) is non-monotonic.

Definition 2.4. A sequence $\{x_n\}$ in a metric space X is said to *converge* if there is a point $x \in X$ with the following property: for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(x_n, x) < \varepsilon$.

In this case, we also say that $\{x_n\}$ converges to x, or that x is the limit of $\{x_n\}$, and we write $x_n \to x$, or

$$\lim_{n\to\infty} x_n = x.$$

If $\{x_n\}$ does not converge, it is said to *diverge*.

We recall that the set of all points x_n ($n = 1, 2, 3, \ldots$) is the *range* of $\{x_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{x_n\}$ is said to be *bounded* if its range is bounded. In the Example 2.3, (a) and (c) are bounded sequences, while (b) and (d) are not.

Example 2.5. Show that $\lim_{n\to\infty}(\frac{2}{3})^n = 0$.

We need to show that for a given $\varepsilon > 0$, after some $n \in J$, the distance between the elements of the sequence and the limit 0 is smaller than ε. In other words, that there exists some N such that for all n larger than N we have $d(x_n, 0) < \varepsilon$. Taking the absolute value, we have $|(\frac{2}{3})^n| < \varepsilon$ for $\forall n \geq N$, and rewriting

$$\begin{aligned} \left(\frac{2}{3}\right)^n &< \varepsilon, \\ \log\left(\frac{2}{3}\right)^n &< \log \varepsilon, \\ n \log\left(\frac{2}{3}\right) &< \log \varepsilon, \\ n &> \frac{\log \varepsilon}{\log 2/3}. \end{aligned}$$

Denote the smallest integer larger than a as $\lceil a \rceil$. Then, one can take $N = \lceil n \rceil$, and for all $n \geq N$, the inequality $n > \frac{\log \varepsilon}{\log 2/3}$ is satisfied. Then, 0 is a limit of $\left(\frac{2}{3}\right)^n$.

Theorem 2.6. *Every bounded, monotonic sequence converges.*

Example 2.7. Show that the sequence

$$x_n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} = \sum_{k=1}^{n} \frac{1}{k!}$$

converges.

To show that the sequence converges, we use the Theorem 2.6, hence, it is sufficient to show that the sequence is monotonic and bounded. To show monotonicity, note that

$$x_{n+1} = x_n + \frac{1}{(n+1)!} > x_n,$$

so $\{x_n\}$ is increasing and hence monotonic. To show that it is bounded, note that

$$\frac{1}{n!} = \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n} = \frac{1}{2 \cdot 3 \cdot \ldots \cdot n} \le \frac{1}{2 \cdot 2 \cdot \ldots \cdot 2} = \frac{1}{2^{n-1}},$$

with strict inequality for $n > 1$. $x_1 = 1$ is finite, hence does not contradict boundedness. For $n > 1$, we have

$$x_n < 1 + \frac{1}{2^1} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} = \frac{1 - (1/2)^n}{1 - 1/2} = 2 - \left(\frac{1}{2}\right)^{n-1} < 2.$$

Because each element of the sequence x_n for $\forall n > 1$ is bounded by 2, the sequence is bounded.

2.1 Limit laws (i)

Corollary 2.8. *Let $\{x_n\}$ and $\{y_n\}$ are convergent sequences, and let c be a constant. Then,*

(a) $\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$.

(b) $\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$.

(c) $\lim_{n\to\infty} c x_n = c \lim_{n\to\infty} x_n$.

(d) $\lim_{n\to\infty} c = c$.

(e) $\lim_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n$.

(f) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ *if* $\lim_{n\to\infty} y_n \neq 0$.

(g) $\lim_{n\to\infty} x_n^p = \left(\lim_{n\to\infty} x_n\right)^p$ *if* $p > 0$ *and* $x_n > 0$.

Example 2.9. Find the limit of $\{x_n\}$, where

$$x_n = \frac{2n^3 + n^2 - 7n}{n^3 + 2n + 2}.$$

Rewrite the nth term of the sequence as

$$\frac{2 + n^{-1} - 7n^{-2}}{1 + 2n^{-2} + 2n^{-3}}.$$

The limit of the numerator and the denominator respectively is

$$\lim_{n\to\infty} \left(2 + \frac{1}{n} - \frac{7}{n^2}\right) = 2, \quad \lim_{n\to\infty} \left(1 + \frac{2}{n^2} + \frac{2}{n^3}\right) = 1,$$

so that $\lim_{n\to\infty} x_n = 2$.

Definition 2.10. Given a sequence $\{x_n\}$, consider a sequence $\{n_k\}$ of natural numbers, such that $n_1 < n_2 < n_3 < \ldots$. Then the sequence $\{x_{n_i}\}$ is called a *subsequence* of $\{x_n\}$. If $\{x_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{x_n\}$.

The sequence $\{x_n\}$ converges to x if and only if every subsequence of $\{x_n\}$ converges to x.

Example 2.11. Consider a sequence $x_n = (-1)^n$ that we know to be divergent. Now, consider two sequences of natural numbers, $\{n_k\} = \{1, 3, 5, \ldots\}$ and $\{m_k\} = \{2, 4, 6, \ldots\}$. The subsequence corresponding to $\{n_k\}$ is $\{-1, -1, -1, \ldots\}$ with the limit -1, and the subsequence corresponding to $\{m_k\}$ is $\{1, 1, 1, \ldots\}$ with the limit 1. Hence, it is possible for subsequences to converge even though the whole sequence does not.

2.2 Upper and lower limits

Definition 2.12. Let $\{x_n\}$ be a sequence of real numbers with the following property: for every real M there is an integer N such that $n \geq N$ implies $x_n \geq M$. We then write

$$x_n \to +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $x_n \leq M$, we write

$$x_n \to -\infty.$$

Definition 2.13. Let $\{x_n\}$ be a sequence or real numbers. Let E be the set of numbers x such that $x_{n_k} \to x$ for some subsequence $\{x_{n_k}\}$. This set E contains all subsequential limits as defined in the Definition 2.10, plus possibly the numbers $+\infty$, $-\infty$.

Put

$$x^\star = \sup E, \quad x_\star = \inf E.$$

The numbers $x^\star$ and $x_\star$ are called the *upper* and *lower limits* of $\{x_n\}$. We use the notation

$$\limsup_{n\to\infty} x_n = x^\star, \quad \liminf_{n\to\infty} x_n = x_\star.$$

Theorem 2.14. *If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then*

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n,$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n.$$

3 Continuity

3.1 Limits of functions

Definition 3.1. Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x\to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y, respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some Euclidean space $\mathbb{R}^k$, the distances d_X, d_Y are of course replaced by absolute values, or by appropriate norms.

Corollary 3.2. *If f has a limit at p, this limit is unique.*

Definition 3.3. One can also define *one-sided (left-sided and right-sided limits)* by manipulating the definition such that it considers not all x in the δ-neighborhood of p but those x that are smaller (or larger) than p:

$$\lim_{x\to p^-} f(x) = q,$$
$$\lim_{x\to p^+} f(x) = q.$$

Theorem 3.4. *It holds that $\lim_{x\to p} f(x) = q$ if and only if $\lim_{x\to p^-} f(x) = \lim_{x\to p^+} f(x) = q$.*

3.2 Limit laws (ii)

Corollary 3.5. *If* $\lim_{x\to p} f(x)$ *and* $\lim_{x\to p} g(x)$ *exist and c is a constant, then*

(a) $\lim\limits_{x\to p}(f(x)+g(x)) = \lim\limits_{x\to p} f(x) + \lim\limits_{x\to p} g(x)$.

(b) $\lim\limits_{x\to p}(f(x)-g(x)) = \lim\limits_{x\to p} f(x) - \lim\limits_{x\to p} g(x)$.

(c) $\lim\limits_{x\to p}(cf(x)) = c\lim\limits_{x\to p} f(x)$.

(d) $\lim\limits_{x\to p} c = c$.

(e) $\lim\limits_{x\to p} x = p$.

(f) $\lim\limits_{x\to p}(f(x)g(x)) = \lim\limits_{x\to p} f(x)\lim\limits_{x\to p} g(x)$.

(g) $\lim\limits_{x\to p}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to p} f(x)}{\lim\limits_{x\to p} g(x)}$ *if* $\lim\limits_{x\to p} g(x) \neq 0$.

(h) $\lim\limits_{x\to p}(f(x))^n = \left(\lim\limits_{x\to p} f(x)\right)^n$, $n\in\mathbb{N}$.

Definition 3.6. We write $f(x)\to+\infty$ as $x\to p$, or

$$\lim_{x\to p} f(x) = +\infty,$$

if for every $\varepsilon>0$, there exists $\delta>0$ such that $f(x)>\varepsilon$ for every x for which $0<|x-p|<\delta$. An example of such a function is $f(x)=x^{-1}$ with a limit $\lim_{x\to 0} f(x)$.

3.3 Continuous functions

Definition 3.7. Suppose X and Y are metric spaces, $E\subset X$, $p\in E$, and f maps E into Y. Then f is said to be *continuous at p* if for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x\in E$ for which $d_X(x,p)<\delta$.

If f is continuous at every point of E, then f is said to be *continuous on E*. It should be noted that f has to be defined at the point p in order to be continuous at p.

We now turn to compositions of functions. A brief statement of the following theorem is that a continuous function of a continuous function is continuous.

Theorem 3.8. *Suppose X, Y, Z are metric spaces,* $E\subset X$, *f maps E into Y, g maps the range of f,* $f(E)$, *into Z, and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \quad (x\in E).$$

If f is continuous at point $p\in E$ *and if g is continuous at the point* $f(p)$, *then h is continuous at p.*

This function h is called the *composition* or the *composite* of f and g. The notation

$$h = g\circ f$$

is frequently used in this context.

Example 3.9. Consider two functions $f(x)=\frac{x}{2}$ and $g(x)=x^2$. We have

(a) $f\circ g = f(g(x)) = \dfrac{g(x)}{2} = \dfrac{x^2}{2}$.

(b) $g\circ f = g(f(x)) = \left(\dfrac{x}{2}\right)^2 = \dfrac{x^2}{4}$.

(c) $g\circ g = g(g(x)) = (x^2)^2 = x^4$.

Theorem 3.10. *Let f and g be functions defined on the same interval. If* $f(x)$ *and* $g(x)$ *are continuous at p, so are* $f(x)+g(x)$ *and* $f(x)g(x)$. *If* $g(p)\neq 0$, $f(x)/g(x)$ *is also continuous at p.*

4 Differentiation

In this section we shall confine our attention to *real* functions defined on intervals or segments.

Definition 4.1. Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t), \tag{1}$$

provided that this limit exists.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit (1) exists; f' is called the *derivative of* f.

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E.

It is possible to consider right-hand and left-hand limits in (1); this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b, the derivative, if it exists, is a right-hand or left-hand derivative respectively.

If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined by (4.1) and (1), as above. But $f'(a)$ and $f'(b)$ are not defined in this case.

Theorem 4.2. *Let f be defined on $[a, b]$. If if is differentiable at a point $x \in [a, b]$, then f is continuous at x.*

Proof. As $t \to x$, we have

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \to f'(x) \cdot 0 = 0.$$

□

The converse of this theorem is not true.

Example 4.3. Consider two functions,

$$f(x) = \begin{cases} x, & x < 0, \\ x^2, & x \geq 0, \end{cases} \quad g(x) = \begin{cases} 0, & x \leq 0, \\ 1 & x > 0. \end{cases}$$

The function $g(x)$ is discontinuous at 0, hence it is not differentiable. The function $f(x)$ is continuous at 0, but not differentiable. To show this, note

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^-} \frac{x - 0}{x} = 1 \neq \lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0^+} \frac{x^2 - 0}{x} = 0.$$

Because one-sided derivatives are not equal, the derivative at 0, $f'(0)$, does not exist.

Theorem 4.4. *Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, $f \cdot g$, and f/g are differentiable at x, and*

(a) $(f + g)'(x) = f'(x) + g'(x)$.

(b) $(f \cdot g)(x) = f'(x)g(x) + f(x)g'(x)$.

(c) $(f/g)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$, $g(x) \neq 0$.

Example 4.5. The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of *(b)* and *(c)* then shows that $f(x) = x^n$ is differentiable, and that its derivative is $f'(x) = nx^{n-1}$, for any integer n. Thus, every polynomial is differentiable and so is every rational function, except at the points where the denominator is zero.

Example 4.6. Consider $f(x) = x^2, g(x) = 1 + x$. Then we have

$$\begin{aligned} f'(x) &= 2x, \\ g'(x) &= 1, \\ (f+g)'(x) &= (x^2+1+x)' = 2x+1, \\ (f \cdot g)'(x) &= (x^2 \cdot (1+x))' = 2x \cdot (1+x) + x^2 = 2x + 3x^2, \\ \left(\frac{f(x)}{g(x)}\right)' &= \frac{2x(1+x) - x^2}{(1+x)^2} = \frac{2x+x^2}{(1+x)^2}. \end{aligned}$$

The following theorem is known as the "chain rule" for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives.

Theorem 4.7. *Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If*

$$h(t) = g(f(t)) \quad (a \le t \le b),$$

then h is differentiable at x, and

$$h'(x) = g'(f(x))f'(x).$$

Example 4.8. Consider two functions, $f(x) = \frac{x}{2}$ and $g(x) = x^2$, and their composite function $h(x) = \left(\frac{x}{2}\right)^2$. Then,

$$\begin{aligned} f'(x) &= \frac{1}{2}, \\ g'(x) &= 2x, \\ h'(x) &= g'(f(x))f'(x) = \frac{x}{2}. \end{aligned}$$

4.1 Mean value theorems

Definition 4.9. Let f be a real function defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \le f(p)$ for all $q \in X$ with $d(p,q) < \varepsilon$.

Local minima are defined likewise. Our next theorem is the basis of many applications of differentiation.

Theorem 4.10. *Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$. The analogous statement for local minima is also true.*

Proof. Choose δ in accordance with Definition 4.9, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \ge 0.$$

Letting $t \to x$, we see that $f'(x) \ge 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \le 0,$$

which shows that $f'(x) \le 0$. Hence, $f'(x) = 0$. □

The following result is usually referred to as the mean value theorem:

Theorem 4.11. *If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which*

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 4.12. *Suppose f is differentiable in (a,b).*

(a) If $f'(x) \ge 0$ for all $x \in (a,b)$, then f is monotonically increasing.

(b) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.

(c) If $f'(x) \le 0$ for all $x \in (a,b)$, then f is monotonically decreasing.

4.2 o and $\mathcal{O}$ notation

Suppose we have a function $f(x)$ with $f(a) = 0$ and we want to consider how quickly the function goes to zero around a. Then ideally, we would want to find a simple function g (for example, $g(x) = (x-a)^n$) which also vanishes at a such that g and f are almost equal around a. The "small-o" and "big-o" notation expresses this notion, but only states that f goes to zero faster than g.

Definition 4.13. We say

$$f(x) = \mathcal{O}(g(x))$$

as $x \to a$ if there exists a constant M such that $|f(x)| \leq M\,|g(x)|$ in some punctured neighborhood of a, that is for $x \in (a-\delta, a+\delta)\backslash\{a\}$ for some value of δ.

We say

$$f(x) = o(g(x))$$

as $x \to a$ if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$. This implies that there exists a punctured neighborhood of a on which g does not vanish.

Example 4.14. The first two examples are derived from Taylor polynomials, the rest can be checked directly:

a) $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \mathcal{O}(x^4)$ as $x \to 0$,

b) $\frac{1}{1-x} = 1 + x + x^2 + \mathcal{O}(x^3) = 1 + x + x^2 + o(x^2)$ as $x \to 0$,

c) $|x^3| = \mathcal{O}(x^3) = o(x^2)$ as $x \to 0$,

d) $\cosh(x) = \mathcal{O}\left(e^x\right) = o\left(e^{\frac{5}{4}x}\right)$ as $x \to 0$,

e) $\frac{1}{\sin(x)} = \mathcal{O}\left(\frac{1}{x}\right) = o\left(\frac{1}{x^{\frac{3}{2}}}\right)$ as $x \to 0$.

Theorem 4.15. *The following holds:*

(a) $f(x) = \mathcal{O}\left(f(x)\right)$.

(b) If $f(x) = o\left(g(x)\right)$ *then* $f(x) = \mathcal{O}\left(g(x)\right)$.

(c) If $f(x) = \mathcal{O}\left(g(x)\right)$ *then* $\mathcal{O}\left(f(x) + g(x)\right) = \mathcal{O}\left(g(x)\right)$.

(d) If $f(x) = \mathcal{O}\left(g(x)\right)$ *then* $o\left(f(x) + g(x)\right) = o\left(g(x)\right)$.

(e) Let $c \neq 0$, *then* $c \cdot \mathcal{O}\left(g(x)\right) = \mathcal{O}\left(g(x)\right)$ *and* $c \cdot o\left(g(x)\right) = o\left(g(x)\right)$.

(f) $\mathcal{O}\left(f(x)\right)\mathcal{O}\left(g(x)\right) = \mathcal{O}\left(f(x)g(x)\right)$.

(g) $o\left(f(x)\right)\mathcal{O}\left(g(x)\right) = o\left(f(x)g(x)\right)$.

(h) If $g(x) = o(1)$ *then* $\frac{1}{1+o(g(x))} = 1 + o\left(g(x)\right)$, *and* $\frac{1}{1+\mathcal{O}(g(x))} = 1 + \mathcal{O}\left(g(x)\right)$.

In the case when functions $f(\cdot)$ and $g(\cdot)$ are polynomials these rules simplify to the following.

Corollary 4.16. *Around 0 we have*

a) $x^a = \mathcal{O}(x^b)$ *for all* $b \leq a$, *and* $x^a = o(x^b)$ *for all* $b < a$.

b) $\mathcal{O}(x^a) + \mathcal{O}(x^b) = \mathcal{O}\left(x^{\min(a,b)}\right)$, $o(x^a) + o(x^b) = o\left(x^{\min(a,b)}\right)$, *and*

$$\mathcal{O}(x^a) + o(x^b) = \begin{cases} o(x^b), & b < a, \\ \mathcal{O}(x^a), & b \geq a. \end{cases}$$

c) For $c \neq 0$, $c \cdot \mathcal{O}(x^a) = \mathcal{O}(x^a)$, *and* $c \cdot o(x^a) = o(x^a)$.

d) $x^b\mathcal{O}(x^a) = \mathcal{O}(x^{a+b})$, *and* $x^b o(x^a) = o(x^{a+b})$.

e) $\mathcal{O}(x^a)\mathcal{O}(x^b) = \mathcal{O}(x^{a+b})$, $\mathcal{O}(x^a)o(x^b) = o(x^{a+b})$, *and* $o(x^a)o(x^b) = o(x^{a+b})$.

4.3 Differentiation of functions of several variables

So far, we have focused on functions of one variable; a straightforward extension of the differentiation ideas to functions of several variables involves *partial derivatives*.

Definition 4.17. Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each x_i at each point $x = (x_1, \ldots, x_n)$ in the domain of f, the *partial derivative* of f at x is

$$\frac{\partial f}{\partial x_i}(x) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

provided that this limit exists.

5 Integration

Definition 5.1. Let f be a function defined on $[a, b]$. Divide the interval $[a, b]$ into n subintervals of equal width, $\Delta x = (b - a)/n$. Let $x_0, x_1, \ldots, x_n$ be the endpoints of these subintervals, and let $x_1^\star, \ldots, x_n^\star$ be any points in these subintervals, so that $x_i^\star$ lies in the ith subinterval $[x_{i-1}, x_i]$. The *definite integral* of f from a to b is

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^\star)\Delta x$$

provided that the limit exists. If it does, we say that f is *integrable* on $[a, b]$.

Sometimes instead of definite integrals, we work with indefinite integrals.

Definition 5.2. *Indefinite integral* (or *antiderivative*) of the function f is defined as

$$\int f(x)dx = F(x),$$

such that $F'(x) = f(x)$.

Note that if $F(x)$ is the antiderivative of $f(x)$, then $F(x) + C$ is also the antiderivative of $f(x)$ for any constant C. Thus, an indefinite integral represents the whole family of functions.

Theorem 5.3. *If f is continuous on $[a, b]$, or if f has only a finite number of jumpdiscontinuities, then f is integrable on $[a, b]$.*

Corollary 5.4. *Let f and g be integrable on $[a, b]$, and k be a constant. Then we have*

(a) $\int_a^b k dx = k(b - a)$.

(b) $\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

(c) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.

(d) $\int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)dx - \int_a^b g(x)dx$.

(e) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$ *for some* $c \in [a, b]$.

(f) $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

(g) $\int_a^b f(x)dx \geq \int_a^b g(x)dx$ *if* $f(x) \geq g(x)$ *for all* $x \in [a, b]$.

Lemma 5.5 (Integration by parts). *Let f and g be integrable, and assume $f'(x)$ and $g'(x)$ exist for all x. Then,*

$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x).$$

For definite integrals defined on $[a,b]$, it holds that

$$\int_a^b f(x)'g(x)dx + \int_a^b f(x)g(x)'dx = (f(x)g(x))\Big|_a^b.$$

Example 5.6. Consider $\int x \sin x dx$. Pick $f(x) = x$, $g(x) = -\cos x$. Then

$$\int x \sin x dx = -x\cos x - \int -\cos x dx = -x\cos x + \sin x + C.$$

Example 5.7. Consider $\int_0^\pi e^x \sin x dx$. First, pick $f(x) = e^x$ and $g(x) = -\cos x$. Then

$$\int_0^\pi e^x \sin x dx = (e^x(-\cos x))\Big|_0^\pi + \int_0^\pi e^x \cos x dx.$$

Let us integrate by parts again. Now pick $f(x) = e^x$ and $g(x) = \sin x$. Then

$$\int_0^\pi e^x \sin x dx = (e^x(-\cos x))\Big|_0^\pi + (e^x \sin x)\Big|_0^\pi - \int_0^\pi e^x \sin x dx.$$

Regrouping, we have

$$\begin{aligned}\int_0^\pi e^x \sin x dx &= \frac{1}{2}\left((e^x(-\cos x))\Big|_0^\pi + (e^x \sin x)\Big|_0^\pi\right)\\ &= \frac{1}{2}\left[e^\pi(-\cos\pi) - e^0(-\cos 0)\right] + \frac{1}{2}\left[e^\pi \sin\pi - e^0 \sin 0\right]\\ &= \frac{e^\pi + 1}{2}.\end{aligned}$$

Lemma 5.8 (Integration by substitution). *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

If g' is continuous on $[a,b]$, and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du.$$

Example 5.9. Consider $\int x^3 \cos(x^4 + 2)dx$. Let $u = x^4 + 2$, then $du = 4x^3 dx$ and $dx = du/4x^3$. So we have

$$\int x^3 \cos(x^2 + 4)dx = \frac{1}{4}\int \cos u du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^2 + 4) + C.$$