Instructor: **Stanislav Anatolyev / Yaroslav Korobka** Discussion Session: **5**
Course: **Dynamic Modeling in Economics I** Date: December 6, 2024

Problem 1

Consider the following state space model:

$$x_t = \mu + y_t + z_t,$$

where y_t and z_t are unobservable processes, y_t following MA(1) with MA-coefficient θ and innovations ε_t with variance σ^2, and z_t following mean-zero stationary AR(1) with AR-coefficient ρ and innovations u_t (uncorrelated with ε_t at all lags and leads) with variance ς^2.

1. Derive an ARMA process for x_t. Write out the process for x_t in a state space form with the state vector $(x_t, x_{t-1})'$ and innovation vector $(\varepsilon_t, u_t)'$. What is this vector process?
2. Google the stationarity/stability condition for vector ARMA processes and verify it for the process under consideration. Intuitively, why does it happen?
3. Google the invertibility condition for vector ARMA processes and show that the process under consideration is never invertible. Intuitively, why does it happen?
4. What is the long-horizon forecast for x_t (that is $\hat{x}_{t+h|t}$ for big h when the transitory dynamics dies out). What is the variance of the unpredictable part?

Problem 2

Consider the pure ARCH(1) model

$$\varepsilon_t = \sigma_t \eta_t,$$

where i.i.d. $\eta_t \sim \mathcal{D}(0, 1)$ for some distribution $\mathcal{D}$ with skewness γ_η and kurtosis κ_η, and

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2,$$

where $\omega > 0$ and $\alpha \geq 0$. Suppose the stationarity conditions hold.

1. Derive[1] the kurtosis coefficient κ_ε of ε_t when it exists. When does it exist?
2. Derive the autocovariance function of the process ε_t^2.

[1] Express it in terms of parameters σ^2, α and κ_η only.

Problem 3

Consider the GARCH(1,1) model

$$\varepsilon_t = \sigma_t \eta_t,$$

where i.i.d. $\eta_t \sim \mathcal{D}(0,1)$ for some distribution $\mathcal{D}$ with skewness γ_η and kurtosis κ_η, and

$$\sigma_t^2 = \omega + \alpha \varepsilon_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where $\omega > 0$, $\alpha > 0$ and $\beta > 0$.

1. Derive the kurtosis coefficient κ_ε of ε_t when it exists. Express it in terms of parameters σ^2, α, β and κ_η only. When does it exist?
2. Recall that the strict stationarity condition for ARCH(1) is $\mathbb{E}\left[\log\left(\alpha\eta_t^2\right)\right] < 0$. Derive the strict stationarity condition for GARCH(1,1).