Exercise session 2

Problem 4

$$y_{it} = \alpha y_{it-1} + \eta_i + v_{it}$$

$$
\begin{aligned}
y_{it} &= \alpha(\alpha y_{it-2} + \eta_i + v_{it-1}) + \eta_i + v_{it} \\
&= \alpha^2 \boxed{y_{it-2}} + \alpha\eta_i + \alpha v_{it-1} + \eta_i + v_{it} \\
&= \alpha^2(\alpha y_{it-3} + \eta_i + v_{it-2}) + \alpha\eta_i + \alpha v_{it-1} + \\
&\quad + \eta_i + v_{it} \\
&= \alpha^3 y_{it-3} + \alpha^2\eta_i + \alpha^2 v_{it-2} + \alpha\eta_i + \alpha v_{it-1} + \\
&\quad + \eta_i + v_{it} \\
&= \dots \\
&= \alpha^t y_{0i} + \left(\sum_{s=0}^{t-1} \alpha^s\right)\eta_i + \left(\sum_{s=0}^{t-1} \alpha^s v_{it-s}\right)
\end{aligned}
$$

Bias is driven by $\boxed{\underbrace{\mathbb{E}[y_{it-1}u_{it}]}}$, $u_{it} := \eta_i + v_{it}$

$$\hat{\alpha} = \frac{\sum_{i=1}^{n} y_{i-}^{*} y_{i}^{*}}{\sum_{i=1}^{n} y_{i-}^{*} y_{i-}^{*\prime}} = \dots = \frac{\qquad}{-\,\|\,-}$$

$$\boxed{y_{it-1}} = \alpha^{t-1} y_{i0} + \underbrace{\left(\sum_{s=0}^{t-2} \alpha^{s}\right)} \eta_i + \left(\sum_{s=0}^{t-2} \alpha^{s} v_{it-s}\right)$$

$$= \alpha^{t-1} y_{i0} + \left(\frac{1-\alpha^{t-1}}{1-\alpha}\right) \eta_i + \left(\sum_{s=0}^{t-2} \alpha^{s} v_{it-s}\right)$$

$$\sum_{k=0}^{n-1} r^{k} = \frac{1-r^{n}}{1-r}.$$

$$\mathbb{E}[y_{it-1} u_{it}] = \underbrace{\mathbb{E}[y_{it-1} \eta_i]}_{*} + \underbrace{\mathbb{E}[y_{it-1} v_{it}]}_{=0}$$

$$* = \mathbb{E}\left[\alpha^{t-1} y_{i0} \eta_i + \left(\frac{1-\alpha^{t-1}}{1-\alpha}\right) \eta_i^{2} + \underbrace{\sum_{s=0}^{t-2} \alpha^{s} \eta_i v_{it-s}}_{\substack{\mathbb{E}[\eta_i v_{it-s}] \\ =0}}\right]$$

$$= \underbrace{\alpha^{t-1} \mathbb{E}[\eta_i y_{i0}]}_{\substack{\text{if assume} \\ =0}} + \boxed{\left(\frac{1-\alpha^{t-1}}{1-\alpha}\right) \sigma_\eta^{2}}$$

$$\sigma_\eta^{2} \neq 0$$

$$\mathbb{E}[\eta_i^{2}] = \mathrm{var}[\eta_i] = \sigma_\eta^{2}$$

Made with Goodnotes

Problem 5

$$y_{it} = \alpha y_{it-1} + \eta_i + v_{it}, \quad \eta_i \sim iid(0, \sigma_\eta^2)$$

$$v_{it} \sim iid(0, \sigma_v^2)$$

$$y_{i0} = 0 \ \forall i$$

Derive $\text{var}[y_{it}]$.

$$y_{it} = \alpha^t y_{i0} + \Big(\sum_{s=0}^{t-1} \alpha^s\Big)\eta_i + \sum_{s=0}^{t-1} v_{it-s}\alpha^s \overset{y_{i0}=0}{=}$$

$$= \Big(\sum_{s=0}^{t-1} \alpha^s\Big)\eta_i + \sum_{s=0}^{t-1} v_{it-s}\alpha^s.$$

- $$\text{var}\Big[\Big(\sum_{s=0}^{t-1} \alpha^s\Big)^2 \eta_i^2\Big] = \Big(\sum_{s=0}^{t-1} \alpha^s\Big)^2 \cdot \sigma_\eta^2 = \sigma_\eta^2 \Big(\frac{1-\alpha^t}{1-\alpha}\Big)^2$$

- $$\text{var}\Big[\sum_{s=0}^{t-1} v_{it-s}^2 \alpha^{2s}\Big] = \sigma_v^2 \sum_{s=0}^{t-1} \alpha^{2s} = \sigma_v^2 \cdot \Big(\frac{1-\alpha^{2t}}{1-\alpha^2}\Big).$$

$$\Rightarrow \text{var}[y_{it}] = \sigma_\eta^2 \Big(\frac{1-\alpha^t}{1-\alpha}\Big)^2 + \sigma_v^2 \cdot \Big(\frac{1-\alpha^{2t}}{1-\alpha^2}\Big).$$

Problem 6

$$y_{it} = \alpha y_{it-1} + \eta_i + \underline{v_{it}}$$

$$\underline{v_{it} = w_{it} + b w_{it-1}}, \quad w_{it} \sim iid(0, \sigma_w^2)$$

Show that y_{it-2} is not a valid instrument, but $y_{it-j}, j \geq 3$ is valid.

Autocovariances $\mathbb{E}[v_{it} v_{it-s}]$:

- $s=0$: $\mathbb{E}[v_{it} v_{it}] = \mathbb{E}[v_{it}^2] = \mathbb{E}[(w_{it} + b w_{it-1})^2]$

$$= \mathbb{E}[w_{it}^2 + 2b w_{it} w_{it-1} + b^2 w_{it-1}^2]$$

$$= \underbrace{\mathbb{E}[w_{it}^2]}_{\sigma} + 2b \underbrace{\mathbb{E}[w_{it} w_{it-1}]}_{} + b^2 \mathbb{E}[w_{it-1}^2]$$

$$= \sigma_w^2 + 0 + b^2 \sigma_w^2 = (1 + b^2)\sigma_w^2.$$

- $\boxed{s=1:}$ $\mathbb{E}[v_{it} v_{it-1}] = \mathbb{E}[(w_{it} + b w_{it-1}) \times (w_{it-1} + b w_{it-2})] =$

$\mathbb{E}[v_{it} v_{it-1}] = \mathbb{E}[v_{it-1} v_{it-2}] = b\sigma_w^2$

$$= \mathbb{E}[\underbrace{w_{it} w_{it-1}}_{0} + \underbrace{w_{it} w_{it-2} \cdot b}_{0} + b w_{it-1}^2 + b w_{it-1} w_{it-2}]$$

$$= \mathbb{E}[w_{it} w_{it-1}] + b\, \mathbb{E}[w_{it} w_{it-2}]$$

$$+ b \underbrace{\mathbb{E}[w_{it-1}^2]}_{} + b^2 \underbrace{\mathbb{E}[w_{it-1} w_{it-2}]}_{0}$$

$$= b\sigma_w^2.$$

$\boxed{s=2:}$ $\boxed{\mathbb{E}[v_{it}v_{it-2}] = 0}$ $= \mathbb{E}[(w_{it}+bw_{it-1})(w_{it-2}+bw_{it-3})]$

$s=3 \qquad = 0$

$s = \ldots$

FD:

$$y_{it} - y_{it-1} = \alpha(y_{it-1} - y_{it-2}) + \underbrace{v_{it} - v_{it-1}}_{:= u_{it}}$$

For y_{it-2} to be valid it should hold that

$$\mathbb{E}[y_{it-2}u_{it}] = 0$$

$$\Leftrightarrow \mathbb{E}[(\ldots + v_{it-2})(v_{it} - v_{it-1})]$$

$$= \ldots = \mathbb{E}[v_{it-1}v_{it-2}] = b\sigma_w^2 \neq 0$$

y_{it-j}, $j \geq 3$: $\mathbb{E}[y_{it-3}u_{it}] = 0$?

$$\mathbb{E}[(\ldots + v_{it-3})(v_{it} - v_{it-1})]$$

$$= \ldots = \mathbb{E}[v_{it-3}v_{it-1}] = \boxed{\mathbb{E}[v_{it-2}v_{it}] = 0}$$

Problem 7

a) WG:

$$\left(\frac{I_{it}}{K_{it}} - \frac{\bar{I}_i}{\bar{K}_i}\right) = \alpha\left(\frac{I_{it-1}}{K_{it-1}} - \frac{\bar{I}_i}{\bar{K}_i}\right) + v_{it} - \bar{v}_i$$

$$\frac{\bar{I}_i}{\bar{K}_i} := \frac{1}{T}\sum_{t=1}^{T}\frac{I_{it}}{K_{it}}.$$

FD:

$$\left(\frac{I_{it}}{K_{it}} - \frac{I_{it-1}}{K_{it-1}}\right) = \alpha\underline{\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right)} + \boxed{v_{it} - v_{it-1}}$$

As instrument, use for example $\frac{I_{it-2}}{K_{it-2}}$, 2SLS

1) $$\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right) = \gamma_0 + \gamma_1\frac{I_{it-2}}{K_{it-2}} + \varepsilon_{i(t)} \sim \text{iid}(0, \sigma_\varepsilon^2)$$

run OLS

$\Rightarrow$

2) $$\left(\frac{I_{it}}{K_{it}} - \frac{I_{it-1}}{K_{it-1}}\right) = \alpha\,\widehat{\underbrace{\left(\frac{I_{it-1}}{K_{it-1}} - \frac{I_{it-2}}{K_{it-2}}\right)}_{:=\hat{\gamma}_1\cdot\left(\frac{I_{it-2}}{K_{it-2}}\right)}} + v_{it} - v_{it-1}$$

$\hat{\gamma}_1$ OLS

Exercise session 3 — Causal inference

Problem 1. Case 1

Table 2: Surgery vs therapy

patient	Y_i^1	Y_i^0	δ_i	Y_i	D_i
		case 1			
1	4	3	1	4	1
1	6	5	1	6	1
3	1	2	−1	2	0
4	3	6	−3	6	0
		case 2			
1	5	3	2	5	1
1	6	5	1	6	1
3	7	6	1	6	0
4	10	8	2	8	0

$$ATE = \frac{1+1-1-3}{4} = -0{,}5$$

$$\rightarrow SDO = \frac{4+6}{2} - \frac{2+6}{2} = 5-4 = 1$$

$$\Rightarrow \boxed{SDO \neq ATE \Rightarrow \text{bias}}$$

$$= 1 = SB + HEB + ATE = 0 + 1{,}5 - 0{,}5$$

1) Selection bias:

$$E[y_i^0 \mid D_i = 1] - E[y_i^0 \mid D_i = 0]$$

$$= \frac{3+5}{2} - \frac{2+6}{2} = 0$$

2) Heterogenous effect bias: $(1-\pi)(ATT - ATU) =$

$$= 0{,}5(1+2) = 1{,}5$$

$$ATT = \mathbb{E}[\delta_i \mid D_i = 1] = 1$$

$$ATU = \mathbb{E}[\delta_i \mid D_i = 0] = \frac{-1-3}{2} = -2$$

$$\pi = \frac{\#\{D_i = 1\}}{N} = \frac{2}{4} = 0{,}5$$

Case 2

$$ATE = \frac{2+1+1+2}{4} = 1{,}5$$

$$SDO = \frac{5+6}{2} - \frac{6+8}{2} = -1{,}5$$

$$ATT = \frac{2+1}{2} = 1{,}5 \qquad ATU = \frac{1+2}{2} = 1{,}5$$

SB: $\mathbb{E}[y_i^0 \mid D_i = 1] - \mathbb{E}[y_i^0 \mid D_i = 0]$

$$= \frac{3+5}{2} - \frac{6+8}{2} = -3$$

HTB: $(1-\pi)(ATT - ATU) =$

$$= 0{,}5(1{,}5 - 1{,}5) = 0$$

Problem 2

$y_i = \beta_0 + \beta_1 D_i + \varepsilon_i$. Show that $\overset{\text{OLS}}{\hat{\beta}_1} = \frac{1}{N_1}\sum_{i=1}^{N_1}(y_i | D_i = 1) - \frac{1}{N_0}\sum_{i=1}^{N_0}(y_i | D_i = 0)$

$$\hat{\beta}_1 = \frac{\textcircled{1}\ \sum_{i=1}^{N}(D_i - \bar{D})(y_i - \bar{y})}{\textcircled{2}\ \sum_{i=1}^{N}(D_i - \bar{D})^2}, \quad \boxed{\bar{D} = \frac{1}{N}\underbrace{\sum_{i=1}^{N} D_i}_{} = \frac{N_1}{N}} \quad D_i \in \{0,1\}$$

$$\sum_{i=1}^{N} D_i = 1 + 1 + 0 \ldots = N_1$$

$$\textcircled{1}\ \sum_{i=1}^{N}(D_i - \bar{D})(y_i - \bar{y}) = \sum_{i=1}^{N} \underline{D_i y_i} - \underline{D_i \bar{y}} - \underline{\bar{D} y_i} + \underline{\bar{D}\bar{y}}$$

- $$\sum_{i=1}^{N} D_i y_i = \frac{N_1}{N_1}\underbrace{\sum_{i=1}^{N} D_i y_i}_{\substack{= 1\cdot y_1 + 1\cdot y_2 + \\ + 0\cdot y_3 + \ldots = \\ = y_1 + y_2 + \ldots}} = \frac{N_1}{N_1}\underbrace{\sum_{\substack{i=1,\\ D_i=1}} y_i}_{} = N_1 \cdot \underbrace{\frac{1}{N_1}\sum_{\substack{i=1\\ D_i=1}} y_i}_{} = N_1 \cdot \bar{y}_1$$

- $$-\sum_{i=1}^{N} \bar{D} y_i = -\sum_{i=1}^{N}\frac{N_1}{N} y_i = -N_1 \cdot \frac{1}{N}\sum_{i=1}^{N} y_i = -N_1 \cdot \bar{y}$$

- $$-\sum_{i=1}^{N} D_i \bar{y} = -\underbrace{\sum_{\substack{i=1\\ D_i=1}} \bar{y}}_{} = -N_1 \cdot \bar{y}$$

$$\Rightarrow \textcircled{1} = N_1\bar{y}_1 - N_1\bar{y} - \cancel{N_1\bar{y}} + \cancel{N_1\bar{y}}$$

$$= N_1(\bar{y}_1 - \bar{y})$$

- $$\sum_{i=1}^{N} \bar{D}\bar{y} = \sum_{i=1}^{N}\frac{N_1}{N}\bar{y} = N_1 \cdot \bar{y}$$

- $N = N_1 + N_0$

HOMEWORK

$$\bar{y} = \frac{1}{N}(N_1\bar{y}_1 + N_0\bar{y}_0) \Rightarrow \boxed{\textcircled{1} = \left(N_1 - \frac{N_1^2}{N}\right)(\bar{y}_1 - \bar{y}_0)}$$

$$② = \sum_{i=1}^{N} (D_i - \bar{D})^2 = \sum_{i=1}^{N} D_i^2 - 2 D_i \bar{D} + \bar{D}^2 =$$

$$= \sum_{i=1}^{N} D_i^2 - 2 \sum_{i=1}^{N} D_i \bar{D} + \sum_{i=1}^{N} \bar{D}^2$$

$$= N_1 - 2 \underbrace{\sum_{i=1}^{N} D_i}\, \frac{N_1}{N} + \sum_{i=1}^{N} \frac{N_1^2}{N^2}$$

$$= N_1 - 2 \frac{N_1^2}{N} + \cancel{N} \cdot \frac{N_1^2}{N^{\cancel{2}}}$$

$$= \boxed{N_1 - \frac{N_1^2}{N}}$$

$$\Rightarrow \hat{\beta} = \frac{\cancel{\left(N_1 - \frac{N_1^2}{N}\right)} (\bar{y}_1 - \bar{y}_0)}{\cancel{N_1 - \frac{N_1^2}{N}}} = \underbrace{\bar{y}_1 - \bar{y}_0}, \quad \bar{y}_1 = \frac{1}{N_1} \sum_{\substack{i=1 \\ D_i = 1}} y_i$$

OLS

$$\bar{y}_0 = \frac{1}{N_0} \sum_{\substack{i=1 \\ D_i = 0}} y_i$$

Made with Goodnotes

Problem 3

$$y_i = \beta_0 + \beta_1 D_i + \varepsilon_i$$

$$\hat{\beta}_1 = \bar{y}_1 - \bar{y}_0 \; ; \; \boxed{\widehat{\text{var}}[\hat{\beta}_1]} = \widehat{\text{var}}[\bar{y}_1] + \widehat{\text{var}}[\bar{y}_0]$$

$$= \boxed{\frac{\hat{\sigma}_1^2}{N_1} + \frac{\hat{\sigma}_0^2}{N_0}}, \quad \sigma_k^2, \; k \in \{0,1\}$$

$$\sigma_1^2 \neq \sigma_0^2$$

Set $x_i := (1 \;\; D_i)'$,

$$\text{var}\left[\begin{pmatrix}\beta_0 \\ \beta_1\end{pmatrix}\right] = \frac{1}{N}\hat{\mathbb{E}}[x_i x_i']^{-1}\mathbb{E}[x_i x_i' \varepsilon_i^2]\,\mathbb{E}[x_i x_i']^{-1} \ldots \text{asympt. variance}$$

$$\Rightarrow \widehat{\text{var}}\left[\begin{pmatrix}\hat{\beta}_0 \\ \hat{\beta}_1\end{pmatrix}\right] = \frac{1}{N}\left(\frac{1}{N}\sum_{i=1}^N x_i x_i'\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^N x_i x_i' \hat{\varepsilon}_i^2\right)\left(\frac{1}{N}\sum_{i=1}^N x_i x_i'\right)^{-1} =$$

$$\hat{\varepsilon}_i = y_i - x_i'\hat{\beta}$$

$$= \frac{1}{N}\underbrace{\left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\right)^{-1}}\left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\hat{\varepsilon}_i^2\right) \times$$

$$\times \left(\frac{1}{N}\sum_{i=1}^N \begin{pmatrix}1 & D_i \\ D_i & D_i\end{pmatrix}\right)^{-1}$$

$$= \frac{1}{N}\begin{pmatrix}1 & N_1/N \\ N_1/N & N_1/N\end{pmatrix}^{-1}\begin{pmatrix}\sum_{i=1}^N \hat{\varepsilon}_i^2/N & \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N \\ \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N & \sum_{i=1, D_i=1} \hat{\varepsilon}_i^2/N\end{pmatrix} \times$$

$$A = \begin{pmatrix}a & b \\ c & d\end{pmatrix} \qquad A^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b \\ -c & a\end{pmatrix}$$

$$\times \begin{pmatrix}1 & N_1/N \\ N_1/N & N_1/N\end{pmatrix}^{-1}$$

$$= \underbrace{\begin{pmatrix}N & N_1 \\ N_1 & N_1\end{pmatrix}^{-1}}\begin{pmatrix}\sum_i \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2 \\ \sum_{i, D_i=1} \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2\end{pmatrix}\underbrace{\begin{pmatrix}N & N_1 \\ N_1 & N_1\end{pmatrix}^{-1}}$$

$$N = N_1 + N_0$$

$$= \underbrace{\begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & -1/N_0 + 1/N_1 \end{pmatrix}}_{A} \times$$

$$\times \underbrace{\begin{pmatrix} \sum_i \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2 \\ \sum_{i, D_i=1} \hat{\varepsilon}_i^2 & \sum_{i, D_i=1} \hat{\varepsilon}_i^2 \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/N_0 + 1/N_1 \end{pmatrix}}_{C}$$

$$\sum_{i=1}^{N} \hat{\varepsilon}_i^2 = \sum_{\substack{i=1 \\ D_i=1}} \hat{\varepsilon}_i^2 + \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2$$

$$\Rightarrow \underbrace{AB}_{D} \times C \Rightarrow DC$$

$$= \begin{pmatrix} \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2 \Big/ N_0 & 0 \\ \sum_{\substack{i=1, \\ D_i=1}} \hat{\varepsilon}_i^2 \Big/ N_1 - \sum_{\substack{i=1 \\ D_i=0}} \hat{\varepsilon}_i^2 \Big/ N_0 & \sum_{\substack{i=1 \\ D_i=1}} \hat{\varepsilon}_i^2 \Big/ N_1 \end{pmatrix} \times$$

$$\times \begin{pmatrix} 1/N_0 & -1/N_0 \\ -1/N_0 & 1/N_0 + 1/N_1 \end{pmatrix} =$$

$$= \begin{pmatrix} \boxed{\sum_{i, D_i=0} \hat{\varepsilon}_i^2 \Big/ N_0^2} & -\sum_{i, D_i=0} \hat{\varepsilon}_i^2 \Big/ N_0^2 \\ -\sum_{i, D_i=0} \hat{\varepsilon}_i^2 \Big/ N_0^2 & \boxed{\dfrac{\sum_{i, D_i=0} \hat{\varepsilon}_i^2}{N_0^2} + \dfrac{\sum_{i, D_i=1} \hat{\varepsilon}_i^2}{N_1^2}} \end{pmatrix}$$

$$= \mathrm{var}\left[\begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix}\right]$$

$$\widehat{\mathrm{var}}[\hat{\beta}_1] = \frac{\sum_{i, D_i=0} \hat{\varepsilon}_i^2}{N_0^2} + \frac{\sum_{i, D_i=1} \hat{\varepsilon}_i^2}{N_1^2}$$

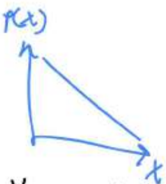
$$\hat{\varepsilon}_i^2 = y_i - x_i' \hat{\beta} \mid D_i = 0 \quad \hat{\sigma}_0^2$$

$$\hat{\varepsilon}_i^2 = y_i - x_i' \hat{\beta} \mid D_i = 1 \quad \hat{\sigma}_1^2$$

es4

$$\underbrace{p(X_i)}_{\text{propensity score}} := \mathbb{P}\{D_i = 1 \mid X_i\} = \mathbb{E}[D_i \mid X_i]$$

	$p(X)$	edu
1	0,7	5
2	0,5	9
3	0,1	10



$$\mathbb{P}\{D=1 \mid \underline{X}, p(X)\} =$$

$$= \mathbb{P}\{D=1 \mid \underline{p}(X)\} = p(X) \Rightarrow X \perp D \mid p(X)$$

Problem 4

i) show that if $p(X)$ is propensity score,

$$X \perp D \mid p(X).$$

- $\mathbb{P}\{D=1 \mid X, p(X)\} = \mathbb{E}[D \mid X, p(X)]$

$$= \mathbb{E}[D \mid X] = \mathbb{P}\{D=1 \mid X\} := p(X)$$

- $\mathbb{P}\{D=1 \mid p(X)\} = \mathbb{E}[D \mid p(X)] = \mathbb{E}[D \mid p(X), X]$

LIE

$$= \mathbb{E}\left[\mathbb{E}[D \mid p(X), X] \mid p(X)\right] = \mathbb{E}\left[\mathbb{E}[D \mid X] \mid p(X)\right]$$

$$= \mathbb{E}[p(X) \mid p(X)] = p(X)$$

ii) if $\underbrace{Y^1, Y^0 \perp D / X}_{1} \Rightarrow Y^1, Y^0 \perp D / p(X)$

$$\mathbb{P}\{D=1 / Y^1, Y^0, p(X)\} = \mathbb{E}[D / Y^1, Y^0, p(X)]$$

$$\overset{\text{LIME}}{=} \mathbb{E}_X \left[\mathbb{E}[D / X, Y^1, Y^0] \,/\, Y^1, Y^0, p(X)\right]$$

$$\overset{1}{=} \mathbb{E}_X \left[\underbrace{\mathbb{E}[D / X]} \,/\, Y^1, Y^0, p(X)\right]$$

$$\overset{\text{ps def.}}{=} \mathbb{E}_X [p(X) / Y^1, Y^0, p(X)] = p(X)$$

$$\Rightarrow \mathbb{P}\{D=1 / \underline{Y^1, Y^0}, \underline{p(X)}\} = \mathbb{P}\{D=1 / \underline{p(X)}\}$$

- homogeneous TE

$y_i^1 - y_i^0 = \delta_i \equiv \delta \ \forall i$

- heterogeneous

$y_i^1 - y_i^0 = \delta_i$

$\delta_i \neq \delta_j, \ i \neq j$

Problem 5.

Show that $\delta_{iv} = \dfrac{\mathrm{cov}[Z_i, Y_i]}{\mathrm{cov}[Z_i, D_i]} \equiv \delta_w = \dfrac{\mathbb{E}[Y_i / Z_i = 1] - \mathbb{E}[Y_i / Z_i = 0]}{\mathbb{E}[D_i / Z_i = 1] - \mathbb{E}[D_i / Z_i = 0]}$

Wald estimand

$y_i = \alpha_0 + \alpha_1 D_i + \varepsilon_i$, $D_i \in \{0,1\}$

$\Rightarrow \mathrm{cov}[D_i, \varepsilon_i] \neq 0$

$\mathrm{cov}[Z_i, \varepsilon_i] = 0$ & $\mathrm{cov}[Z_i, D_i] \neq 0$

LATE ✓

monotonicity

		$Z_i = 1$	
		$D_i = 1$	$D_i = 0$
$Z_i = 0$	$D_i = 1$	always takers	~~defiers~~
	$D_i = 0$	compliers	never takers

Numerator:

- $\mathrm{cov}[Z_i, Y_i] = \mathbb{E}[Z_i Y_i] - \mathbb{E}[Z_i]\mathbb{E}[Y_i]$

$= \mathbb{E}[\mathbb{E}[Z_i Y_i | Z_i = 1]] - \mathbb{E}[Z_i]\mathbb{E}[Y_i]$

$= \mathbb{E}[\mathbb{P}\{Z_i = 1\} \cdot \mathbb{E}[Y_i | Z_i = 1]] - \mathbb{E}[Z_i]\mathbb{E}[Y_i]$

$= \mathbb{P}\{Z_i = 1\}\mathbb{E}[Y_i | Z_i = 1] - \mathbb{E}[Z_i]\mathbb{E}[Y_i]$

$= \mathbb{P}\{Z_i = 1\}\mathbb{E}[Y_i | Z_i = 1] - \mathbb{P}\{Z_i = 1\}\Big(\mathbb{E}[Y_i | Z_i = 1] \cdot \mathbb{P}\{Z_i = 1\} + \mathbb{E}[Y_i | Z_i = 0] \cdot \underbrace{\mathbb{P}\{Z_i = 0\}}_{:= 1 - \mathbb{P}\{Z_i = 1\}}\Big) =$

$= \big(\mathbb{E}[Y_i | Z_i = 1] - \mathbb{E}[Y_i | Z_i = 0]\big) \times \mathbb{P}\{Z_i = 1\} \times (1 - \mathbb{P}\{Z_i = 1\}).$

Denominator:

$\mathrm{cov}[Z_i, D_i] =$

$= \big(\mathbb{E}[D_i | Z_i = 1] - \mathbb{E}[D_i | Z_i = 0]\big) \times \mathbb{P}\{Z_i = 1\}(1 - \mathbb{P}\{Z_i = 1\})$

$$\mathbb{E}[Y_i \mid Z_i = 0] = \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) D_i \mid Z_i = 0]$$
$$= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) \mid D_i = 1, Z_i = 0] \underbrace{\mathbb{P}\{D_i = 1 \mid Z_i = 0\}}_{=0}$$
$$= \mathbb{E}[Y_i^0].$$

Problem 6

Instead of monotonicity, we assume $\mathbb{P}\{D_i = 1 \mid Z_i = 0\} = 0$. (eligibility rule)

Show that we can identify ATT.

$$Y_i = Y_i^0 + (Y_i^1 - Y_i^0) D_i$$
$$\Rightarrow \begin{cases} Y_i = Y_i^1, D_i = 1 \\ Y_i = Y_i^0, D_i = 0 \end{cases}$$

$$\mathbb{E}[Y_i \mid Z_i = 1] = \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) D_i \mid Z_i = 1]$$
$$= \mathbb{E}[Y_i^0] + \mathbb{E}[\mathbb{E}[Y_i^1 - Y_i^0] D_i \mid D_i = 1, Z_i = 1]]$$
$$= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) \mid D_i = 1, Z_i = 1]\, \mathbb{P}\{D_i = 1 \mid Z_i = 1\}$$
$$= \mathbb{E}[Y_i^0] + \mathbb{E}[(Y_i^1 - Y_i^0) \mid D_i = 1] \underbrace{\mathbb{P}\{D_i = 1 \mid Z_i = 1\}}_{\neq 1}.$$

$$ATT = \mathbb{E}[Y_i^1 - Y_i^0 \mid D_i = 1]$$

Because $D_i = 1$ is sufficient for $Z_i = 1$

CS5

Problem 7

$$y_i = \alpha_1 + \beta D_i + u_i,$$
$$\mathbb{E}[D_i u_i] = 0$$

ITT: $y_i = \alpha + \gamma Z_i + e_i$

(intention-to-treat) $\gamma = \mathbb{E}[y_i \mid Z_i = 1] - \mathbb{E}[y_i \mid Z_i = 0]$

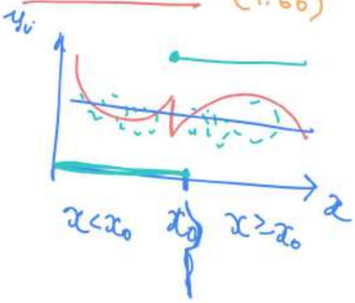
	pr. death	pr. suicide
$\hat{\gamma}$ = Difference	0.02%	0.03%

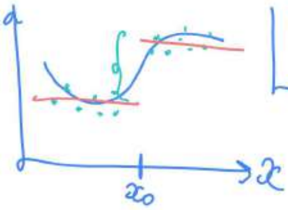
IV: $$\delta_w = \frac{\mathbb{E}[y_i \mid Z_i = 1] - \mathbb{E}[y_i \mid Z_i = 0]}{\underbrace{\mathbb{E}[D_i \mid Z_i = 1] - \mathbb{E}[D_i \mid Z_i = 0]}_{0.1362}}$$

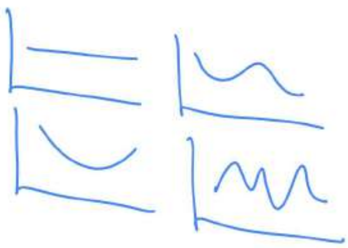
$$= \begin{cases} \dfrac{0.0002}{0.1362} = \underset{(0.0037)}{0.0015}, & \text{death} \\ \dfrac{0.0003}{0.1362} = \underset{(0.0016)}{0.0022}, & \text{suicide} \end{cases}$$

Problem 8

$\underset{(1.66)}{-1.62}$







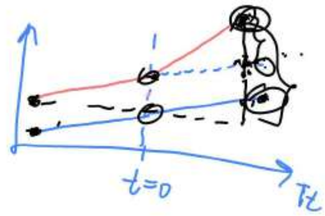
$$\Rightarrow \beta = \left(\mathbb{E}[y_{it} \mid D_i=1, T_t=1] - \mathbb{E}[y_{it} \mid D_{it}=1, T_t=0]\right)$$
$$- \left(\mathbb{E}[y_{it} \mid D_i=0, T_t=1] - \mathbb{E}[y_{it} \mid D_i=0, T_t=0]\right)$$

Problem 9

$$y_{it} = \beta_0 + \beta_D D_i + \beta_T T_t + \beta D_i T_t + u_{it}, \quad \mathbb{E}[u_{it} \mid D_i, T_t] = 0$$

Show that

$$\beta = \left(\mathbb{E}[y_{it} \mid D_i=1, T_t=1] - \mathbb{E}[y_{it} \mid D_i=1, T_t=0]\right)$$
$$- \left(\mathbb{E}[y_{it} \mid D_i=0, T_t=1] - \mathbb{E}[y_{it} \mid D_i=0, T_t=0]\right).$$



- $\mathbb{E}[y_{it} \mid D_i=1, T_t=1] = \beta_0 + \beta_D + \beta_T + \beta \Rightarrow \beta = \mathbb{E}[y_{it} \mid D_i=1, T_t=1] - (\beta_0 + \beta_D) - \beta_T \Rightarrow$
- $\mathbb{E}[y_{it} \mid D_i=1, T_t=0] = \beta_0 + \beta_D$
- $\mathbb{E}[y_{it} \mid D_i=0, T_t=1] = \beta_0 + \beta_T$ $\Big\} \Rightarrow \beta_T = \mathbb{E}[y_{it} \mid D_i=0, T_t=1] - \mathbb{E}[y_{it} \mid D_i=0, T_t=0]$
- $\mathbb{E}[y_{it} \mid D_i=0, T_t=0] = \beta_0$

Problem 10

(1) $y_{its} = \alpha + \gamma NJ_s + \lambda_t dt + \beta (NJ_s \cdot dt) + \varepsilon_{ist}$

(2) $y_{its} = \gamma_s + \theta_t + \beta D_{st} + u_{ist}$

(1): $NJ_s = \begin{cases} 1 & \text{if } i \text{ is in New-Jersey} \\ 0 & \text{if } i \text{ is in Penn} \end{cases}$

$dt = \begin{cases} 1 & \text{if } i \text{ in November} \\ 0 & \text{if } i \text{ in February} \end{cases}$

(1):

- $\mathbb{E}[y_{its} / S = PA, t = Feb] = \alpha$
- $\mathbb{E}[y_{its} / s = PA, t = Nov] = \alpha + \lambda t$
- $\mathbb{E}[y_{its} / s = NJ, t = Feb] = \alpha + \gamma$
- $\mathbb{E}[y_{its} / s = NJ, t = Nov] = \alpha + \gamma + \lambda t + \beta$

(2): $\gamma_s = \begin{cases} \gamma_{PA} & \text{for Penn} \\ \gamma_{NJ} & \text{for New-Jersey} \end{cases}$

$\theta_t = \begin{cases} \theta_{NOV} & \text{for November} \\ \theta_{FEB} & \text{for February} \end{cases}$

$D_{st} \begin{cases} = 1 & \text{if } i \text{ is in New-Jersey in November} \\ = 0 & \text{o/w} \end{cases}$

$$\Rightarrow \lambda_t = \mathbb{E}[y_{its} \mid s = PA,\ t = Nov] - \mathbb{E}[y_{its} \mid s = PA,\ t = Feb]$$

$$\gamma = \mathbb{E}[y_{its} \mid s = NJ,\ t = Feb] - \mathbb{E}[y_{its} \mid s = PA,\ t = Feb]$$

$$\beta = \left(\mathbb{E}[y_{its} \mid s = NJ,\ t = Nov] - \mathbb{E}[y_{its} \mid s = NJ,\ t = Feb]\right) - \left(\mathbb{E}[y_{its} \mid s = PA,\ t = Nov] - \mathbb{E}[y_{its} \mid s = PA,\ t = Feb]\right)$$

(2): • $\mathbb{E}[y_{its} \mid s = PA,\ t = Feb] = \gamma_{PA} + \theta_{FEB}$

• $\mathbb{E}[y_{its} \mid s = PA,\ t = Nov] = \gamma_{PA} + \theta_{Nov}$

• $\mathbb{E}[y_{its} \mid s = NJ,\ t = Feb] = \gamma_{NJ} + \theta_{Feb}$

• $\mathbb{E}[y_{its} \mid s = NJ,\ t = Nov] = \gamma_{NJ} + \theta_{Nov} + \beta$

$\Longrightarrow$

$$\alpha = \gamma_{PA} + \theta_{FEB}$$

$$\gamma = \gamma_{NJ} + \theta_{FEB} - \gamma_{PA} - \theta_{FEB} = \gamma_{NJ} - \gamma_{PA}$$

$$\lambda_t = \gamma_{PA} + \theta_{Nov} - \gamma_{PA} - \theta_{FEB} = \theta_{Nov} - \theta_{FEB}$$

$$\beta = \mathbb{E}[y_{its} \mid s = NJ, t = Nov] - \gamma_{NJ} - \theta_{Nov}$$

$$= (\mathbb{E}[y_{its} \mid s = NJ, t = Nov] - \mathbb{E}[y_{its} \mid s = PA, t = Nov])$$

$$- \underbrace{(\gamma_{NJ} - \gamma_{PA})}_{\substack{\gamma = \mathbb{E}[y_{its} \mid s = NJ, t = Feb] - \\ - \mathbb{E}[y_{its} \mid s = PA, t = Nov]}}$$

$$\Rightarrow \beta_{(1)} = \beta_{(2)} = \beta.$$