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Course: **Statistics**

1 Basic probability theory

Exercise 1.1. For each of the following experiments, describe the sample space.

(a) Toss a coin four times.

(b) Count the number of insect-damaged leaves on a plant.

(c) Measure the lifetime (in hours) of a particular brand of light bulb.

(d) Record the weights of 10-day-old rats.

(e) Observe the proportion of defectives in a shipment of electronic components.

Solution. The sample space S is the set of all possible outcomes.

(a) For each coin toss in the series we receive either Head (H) or Tail (T). The example of the realization of the series of four tosses: THHH, meaning 1st toss gives T and the remaining tosses give H. The sample space S is the set of all such combinations. One may want to calculate the cardinality of this set. It is $2^4 = 16$: on each position (toss) we have either T or H (2 options) and we observe four positions.

(b) The number of leaves should be a nonnegative integer. Thus, $S = \{0, 1, 2, 3, \ldots\}$.

(c) The lifetime may be less than an hour or some integer number of hours plus the remaining fraction of an hour. Moreover, the lifetime cannot be negative. Thus, $S = \{0, 1, 2, 3, \ldots\}$ in case the lifetime is measured in hours, and $S = \{t : t \geq 0\}$ in case the lifetime is measured in infinitesimal units of time.

(d) We need to choose measurement units. Consider grams, then we may say $S = (0, +\infty)$. We may also define some upper bound, for example 1 kilogram. Then $S = (0, 1000]$.

(e) One can define n to be the number of items in the shipment. Since we need a proportion, the sample space is $S = \{\frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}\}$.

Exercise 1.2. Verify the following identities:

(a) $A \setminus B = A \setminus (A \cap B) = A \cap B^c$,

(b) $B = (B \cap A) \cup (B \cap A^c)$,

(c) $B \setminus A = B \cap A^c$,

(d) $A \cup B = A \cup (B \cap A^c)$.

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Solution. When working with set expressions one may use Venn diagrams to understand how to proceed. However, the illustration does not constitute a formal proof and thus is not sufficient as an answer. To proof that the expression is true one must explicitly show both directions: if $x \in$ LHS (left-hand side) then $x \in$ RHS (right-hand side) and if $x \in$ RHS then $x \in$ LHS.

(a) $A \setminus B \iff x \in A$ and $x \notin B \iff x \in A$ and $x \notin A \cap B \iff x \in A \setminus (A \cap B)$. At the same time, $x \in A$ and $x \notin B \iff x \in A$ and $x \in B^c \iff x \in A \cap B^c$.

(b) By definition $A \cup A^c = S$. Then using the Distributive Law $(B \cap A) \cup (B \cap A^c) = B \cap (A \cup A^c) = B \cap S = B$. One may also show the equality by the technique used in point (a).

(c) $B \setminus A \iff x \in B$ and $x \notin A \iff x \in B$ and $x \in A^c \iff x \in B \cap A^c$.

(d) $A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = A \cup (B \cap A) \cup A \cup (B \cap A^c) = A \cup [A \cup (B \cap A^c)] = A \cup (B \cap A^c)$.

Exercise 1.3. For events A and B, find formulas for the probabilities of the following events in terms of the quantities $\mathbb{P}(A)$, $\mathbb{P}(B)$, and $\mathbb{P}(A \cap B)$:

(a) either A or B or both,

(b) either A or B but not both,

(c) at least one of A or B,

(d) at most one of A or B.

Solution.

(a) "either A or B or both" means $A \cup B$, so we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

This is because $A \cup B = A \cup (B \cup A^c)$ (Exercise 1.2. point (d)), from it $\mathbb{P}(A \cup B) = \mathbb{P}(A \cup (B \cup A^c)) = \mathbb{P}(A) + \mathbb{P}(B \cap A^c) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.

In the last expression, the second equality uses the fact that A and $B \cap A^c$ are disjoint; the last equality comes from rearranging the following $\mathbb{P}(B) = \mathbb{P}((B \cap A) \cup (B \cap A^c)) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c)$ (combines Exercise 1.2. point (b) and the fact that two sets are disjoint).

It is easier to draw the picture and note the fact that we count the area of the intersection two times if we do not subtract $\mathbb{P}(A \cap B)$.

(b) "either A or B but not both" is $(A \cap B^c) \cup (B \cap A^c)$, so we have

$$\begin{aligned}\mathbb{P}((A \cap B^c) \cup (B \cap A^c)) &= \mathbb{P}(A \cap B^c) + \mathbb{P}(B \cap A^c)\\ &= [\mathbb{P}(A) - \mathbb{P}(A \cap B)] + [\mathbb{P}(B) - \mathbb{P}(B \cap A)]\\ &= \mathbb{P}(A) + \mathbb{P}(B) - 2\mathbb{P}(A \cap B).\end{aligned}$$

The easy way is to draw a picture and note that this probability is equal to $\mathbb{P}(A \cup B) - \mathbb{P}(A \cap B)$.

(c) "at least one of A or B" is $A \cup B$, so we have

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(d) "at most one of A or B" is $(A \cap B)^c$, so we have

$$\mathbb{P}((A \cap B)^c) = 1 - \mathbb{P}(A \cap B).$$

Note that this event also includes the possibility of not A and not B.

Exercise 1.4. Consider two different setups:

(a) A fair dice is cast until a 6 appears. What is the probability that it must be cast more than five times?

(b) Prove that if $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, then:

- if A and B are mutually exclusive, they cannot be independent,
- if A and B are independent, they cannot be mutually exclusive.

Solution.

(a) It must be cast more than five times if we did not observe the appearance of 6 on first five casts. We do not observe 6 on each cast with probability $5/6$. Since casts of a die are independent (adequate additional interpretation of the setup which you should explicitly mention in your solution) the probability to not receive 6 in first five casts is $(5/6)^5 \approx 0.4$.

(b)
- Let A and B be mutually exclusive. Suppose, for the sake of contradiction, that A and B are independent, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. However, from the initial property $A \cap B = \emptyset$ and $\mathbb{P}(A \cap B) = 0$ (mutually exclusive). Since we are given events A and B with positive probabilities, we come to a contradiction $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) = 0$ with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Thus, mutually exclusive events cannot be independent.
- Independence of A and B together with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$ gives $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) > 0$. It means that $A \cap B \neq \emptyset$. Therefore, A and B are not mutually exclusive by definition. Thus, independent events cannot be mutually exclusive.

  One may also proceed by contradiction: let A and B be independent. Suppose, for the sake of contradiction, that A and B are mutually exclusive, then $\mathbb{P}(A \cap B) = 0$. However, from independence and $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$ we have $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) > 0$. This is a desired contradiction.

Exercise 1.5. Two coins, one with $\mathbb{P}(\text{head}) = u$ and one with $\mathbb{P}(\text{head}) = w$, are to be tossed together independently. Let

$$p_0 = \mathbb{P}(0 \text{ heads occur}), \quad p_1 = \mathbb{P}(1 \text{ heads occur}), \quad p_2 = \mathbb{P}(2 \text{ heads occur}).$$

Can u and w be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

Solution. Start with probabilities: $p_0 = (1-u)(1-w)$, that is, first is T and second is T; $p_1 = (1-u)w + u(1-w)$, that is, first is T and second is H or first is H and second is T; and $p_2 = uw$, that is, both are H. Equating these probabilities, we receive

$$p_0 = p_2 \Rightarrow u + w = 1, \quad p_1 = p_2 \Rightarrow uw = \frac{w+u}{3}.$$

These together imply

$$u(1-u) = \frac{1}{3}.$$

This equation has no solution in the real numbers (more specifically, the right-hand side must be $\leq \frac{1}{4}$ for the solution in real numbers to exist). Thus, we cannot choose legitimate u and w to satisfy the conditions.

Exercise 1.6. Consider telegraph signals "dot" and "dash" sent in the proportion 3:4, where erratic transmissions cause a dot to become a dash with probability $1/4$ and a dash to become a dot with probability $1/3$.

(a) If a dash is received, what is the probability that a dash has been sent? If a dot is received, what is the probability that a dot has been sent?

(b) Assuming independence between signals, if the message dot-dot was received, what is the probability distribution of the four possible messages that could have been sent?